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THE COMBINATORICS OF AL-SALAM-CHIHARA q -LAGUERRE POLYNOMIALS

ANISSE KASRAOUI, DENNIS STANTON, AND JIANG ZENG

ABSTRACT. We describe various aspects of the Al-Salam-Chihara q -Laguerre polynomials. These include combinatorial descriptions of the polynomials, the moments, the orthogonality relation and a combinatorial interpretation of the linearization coefficients.

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MR Subject Classifications: Primary 05A18; Secondary 05A15, 05A30.

1. INTRODUCTION

The monic simple Laguerre polynomials $L_n(x)$ may be defined by the explicit formula:

$$L_n(x) = \sum_{k=0}^n (-1)^{n-k} \frac{n!}{k!} \binom{n}{k} x^k, \quad (1)$$

or by the three-term recurrence relation

$$L_{n+1}(x) = (x - (2n + 1))L_n(x) - n^2 L_{n-1}(x). \quad (2)$$

The moments are

$$\mu_n = \mathcal{L}(x^n) = \int_0^\infty x^n e^{-x} dx = n!. \quad (3)$$

The linearization formula reads as follows:

$$L_{n_1}(x)L_{n_2}(x) = \sum_{n_3} C_{n_1 n_2}^{n_3} L_{n_3}(x),$$

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where

$$C_{n_1 n_2}^{n_3} = \sum_{s \geq 0} \frac{n_1! n_2! 2^{N_2+n_3-2s} s!}{(s-n_1)!(s-n_2)!(s-n_3)!(N_2+n_3-2s)!n_3!}.$$

Equivalently we have

$$\mathcal{L}(L_{n_1}(x)L_{n_2}(x)L_{n_3}(x)) = \sum_{s \geq 0} \frac{n_1! n_2! n_3! 2^{N_2+n_3-2s} s!}{(s-n_1)!(s-n_2)!(s-n_3)!(N_2+n_3-2s)!}. \quad (4)$$

Given positive integers n_1, n_2, \dots, n_k such that $n = n_1 + \dots + n_k$, let S_i be the consecutive integer segment $\{n_1 + \dots + n_{i-1} + 1, \dots, n_1 + \dots + n_i\}$ with $n_0 = 0$, then $S_1 \cup \dots \cup S_k = [n]$. A permutation σ of $[n]$ is said to be a *generalized derangement* if i and $\sigma(i)$ do not belong to a same segment S_j for all $i \in [n]$. Let \mathcal{D}_n be the set of generalized derangements of $[n]$ then we have

$$\mathcal{L}(L_{n_1}(x) \dots L_{n_k}(x)) = \sum_{\sigma \in \mathcal{D}_n} 1. \quad (5)$$

A q -version of (1) was studied by Garsia and Remmel [9] in 1980. Several q -analogues of the moments (2) and recurrence relation (3) were investigated in the last two decades (see [2, 18, 19]) in order to obtain new *mahonian* statistics on the symmetric groups. On the other hand, in view of the unified combinatorial interpretations of several aspects of Sheffer orthogonal polynomials (moments, polynomials, and the linearization coefficients) (see [14, 20, 22]) it is natural to seek for a q -version of this picture.

As one can expect, the first result in this direction was the linearization formula for q -Hermite polynomials due to Ismail, Stanton and Viennot [12], dated back to 1987. In particular, their formula provides a combinatorial evaluation of the Askey-Wilson integral. However, a similar formula for q -Charlier polynomials was discovered only recently by Anshelevich [1], who used the machinery of q -Levy stochastic processes. Short later, Kim, Stanton and Zeng [15] gave a combinatorial proof of Anshelevich's result.

The object of this paper is to give a q -version of all the above formulas for simple Laguerre polynomials.

2. AL-SALAM-CHIHARA POLYNOMIALS REVISITED

The Al-Salam-Chihara polynomials $Q_n(x) := Q_n(x; \alpha, \beta | q)$ may be defined by the recurrence relation [16, Chapter 3]:

$$\begin{cases} Q_0(x) = 1, & Q_{-1}(x) = 0, \\ Q_{n+1}(x) = (2x - (\alpha + \beta)q^n)Q_n(x) - (1 - q^n)(1 - \alpha\beta q^{n-1})Q_{n-1}(x), & n \geq 0. \end{cases} \quad (6)$$

Let $Q_n(x) = 2^n p_n(x)$ then

$$xp_n(x) = p_{n+1}(x) + \frac{1}{2}(\alpha + \beta)q^n p_n(x) + \frac{1}{4}(1 - q^n)(1 - \alpha\beta q^{n-1})p_{n-1}(x). \quad (7)$$

They also have the following explicit expressions:

$$\begin{aligned} Q_n(x; \alpha, \beta|q) &= \frac{(\alpha\beta; q)_n}{\alpha^n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, \alpha u, \alpha u^{-1} \\ \alpha\beta, 0 \end{matrix} \middle| q; q \right) \\ &= (\alpha u; q)_n u^{-n} {}_2\phi_1 \left(\begin{matrix} q^{-n}, \beta u^{-1} \\ \alpha^{-1} q^{-n+1} u^{-1} \end{matrix} \middle| q; \alpha^{-1} q u \right) \\ &= (\beta u^{-1}; q)_n u^n {}_2\phi_1 \left(\begin{matrix} q^{-n}, \alpha u \\ \beta^{-1} q^{-n+1} u \end{matrix} \middle| q; \beta^{-1} q u^{-1} \right), \end{aligned}$$

where $x = \frac{u+u^{-1}}{2}$ or $x = \cos \theta$ if $u = e^{i\theta}$.

The Al-Salam-Chihara polynomials have the following generating function

$$G(t, x) = \sum_{n=0}^{\infty} Q_n(x; \alpha, \beta|q) \frac{t^n}{(q; q)_n} = \frac{(\alpha t, \beta t; q)_{\infty}}{(te^{i\theta}, te^{-i\theta}; q)_{\infty}}.$$

They are orthogonal with respect to the linear functional $\hat{\mathcal{L}}_q$:

$$\hat{\mathcal{L}}_q(x^n) = \frac{1}{2\pi} \int_0^{\pi} (\cos \theta)^n \frac{(q, \alpha\beta, e^{2i\theta}, e^{-2i\theta}; q)_{\infty}}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, \beta e^{i\theta}, \beta e^{-i\theta}; q)_{\infty}} d\theta, \quad (8)$$

where $x = \cos \theta$. Note that

$$\hat{\mathcal{L}}_q(Q_n(x)^2) = (q; q)_n (\alpha\beta; q)_n.$$

Theorem 1. *We have*

$$Q_{n_1}(x)Q_{n_2}(x) = \sum_{n_3 \geq 0} C_{n_1, n_2}^{n_3}(\alpha, \beta; q) Q_{n_3}(x), \quad (9)$$

where

$$\begin{aligned} C_{n_1, n_2}^{n_3}(\alpha, \beta; q) &= (-1)^{N_2+n_3} \frac{(q; q)_{n_1} (q; q)_{n_2}}{(\alpha\beta; q)_{n_3}} \\ &\times \sum_{m_2, m_3} \frac{(\alpha\beta; q)_{n_1+m_3} \alpha^{m_2} \beta^{n_3+n_2-n_1-m_2-2m_3} q^{\binom{m_2}{2} + (n_3+n_2-n_1-m_2-2m_3)}}{(q; q)_{n_3+n_2-n_1-m_2-2m_3} (q; q)_{m_2} (q; q)_{m_3+n_1-n_3} (q; q)_{m_3+n_1-n_2} (q; q)_{m_3}}. \end{aligned}$$

Proof. Clearly $C_{n_1, n_2}^{n_3}(\alpha, \beta; q) = \hat{\mathcal{L}}_q(Q_{n_1}(x)Q_{n_2}(x)Q_{n_3}(x))/\hat{\mathcal{L}}_q(Q_{n_3}(x)Q_{n_3}(x))$. Using the Askey-Wilson integral:

$$\frac{(q; q)_{\infty}}{2\pi} \int_0^{\pi} \frac{(e^{2i\theta}, e^{-2i\theta}; q)_{\infty}}{\prod_{j=1}^4 (t_j e^{i\theta}, t_j e^{-i\theta}; q)_{\infty}} d\theta = \frac{(t_1 t_2 t_3 t_4; q)_{\infty}}{\prod_{1 \leq j < k \leq 4} (t_j t_k; q)_{\infty}},$$

one can prove [12, Theorem 3.5] that

$$\begin{aligned} \hat{\mathcal{L}}_q(G(t_1, x)G(t_2, x)G(t_3, x)) \\ = \frac{(\alpha t_1 t_2 t_3, \beta q t_1 t_2 t_3, \alpha\beta q; q)_{\infty}}{(t_1 t_2, t_1 t_3, t_2 t_3; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} t_1 t_2, & t_1 t_3, & t_2 t_3 \\ \alpha t_1 t_2 t_3, & \beta t_1 t_2 t_3 \end{matrix} \middle| q; \alpha\beta \right). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n_1, n_2, n_3} \hat{\mathcal{L}}_q(Q_{n_1}(x)Q_{n_2}(x)Q_{n_3}(x)) \frac{t_1^{n_1}}{(q; q)_{n_1}} \frac{t_2^{n_2}}{(q; q)_{n_2}} \frac{t_3^{n_3}}{(q; q)_{n_3}} \\ = \sum_{k \geq 0} \frac{(\alpha t_1 t_2 t_3 q^k, \beta t_1 t_2 t_3 q^k, \alpha \beta; q)_\infty}{(t_1 t_2 q^k, t_1 t_3 q^k, t_2 t_3 q^k; q)_\infty} \frac{(\alpha \beta)^k}{(q; q)_k}. \end{aligned} \quad (10)$$

Using the Euler formulas:

$$(t; q)_\infty = \sum_{n \geq 0} \frac{(-1)^n q^{\binom{n}{2}}}{(q; q)_n} t^n; \quad \frac{1}{(t; q)_\infty} = \sum_{n \geq 0} \frac{1}{(q; q)_n} t^n,$$

we can rewrite the sum in (10) as follows:

$$\begin{aligned} (\alpha \beta; q)_\infty \sum_{k \geq 0} \frac{(\alpha \beta)^k}{(q; q)_k} \sum_{l_1, l_2 \geq 0} \frac{\alpha^{l_1} \beta^{l_2} q^{k(l_1+l_2)} (-t_1 t_2 t_3)^{l_1+l_2} q^{\binom{l_1}{2} + \binom{l_2}{2}}}{(q; q)_{l_1} (q; q)_{l_2}} \\ \times \sum_{m_1, m_2, m_3 \geq 0} \frac{q^{(m_1+m_2+m_3)k} t_1^{m_1+m_2} t_2^{m_1+m_3} t_3^{m_1+m_3}}{(q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3}}. \end{aligned} \quad (11)$$

Substituting

$$\sum_{k \geq 0} \frac{(\alpha \beta q^{l_1+l_2+m_1+m_2+m_3})^k}{(q; q)_k} = \frac{1}{(\alpha \beta q^{l_1+l_2+m_1+m_2+m_3}; q)_\infty}$$

in (11), we get

$$\sum_{l_1, l_2, m_1, m_2, m_3} t_1^{n_1} t_2^{n_2} t_3^{n_3} \frac{(\alpha \beta)_{n_1+m_3} \alpha^{l_1} \beta^{l_2} q^{\binom{l_1}{2} + \binom{l_2}{2}}}{(q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3} (q; q)_{l_1} (q; q)_{l_2}} (-1)^{l_1+l_2}, \quad (12)$$

where $l_1 + l_2 + m_1 + m_2 = n_1$, $l_1 + l_2 + m_1 + m_3 = n_2$ and $l_1 + l_2 + m_2 + m_3 = n_3$.

Since $l_1 + l_2 \equiv N_2 + n_3 \pmod{2}$, extracting the coefficient of $\frac{t_1^{n_1} t_2^{n_2} t_3^{n_3}}{(q; q)_{n_1} (q; q)_{n_2} (q; q)_{n_3}}$ in (12) and dividing by $(q, \alpha \beta; q)_{n_3}$ we obtain (9) where l_1 is replaced by m_2 . \square

3. THE NEW q -LAGUERRE POLYNOMIALS

We define the new q -Laguerre polynomials $L_n(x; q)$ by re-scaling Al-Salam-Chihara polynomials:

$$L_n(x; q) = \left(\frac{\sqrt{y}}{q-1} \right)^n Q_n \left(\frac{(q-1)x + y + 1}{2\sqrt{y}}; \frac{1}{\sqrt{y}}, \sqrt{y}q|q \right). \quad (13)$$

It follows from (7) that the polynomials $L_n(x; q)$ satisfy the recurrence:

$$L_{n+1}(x; q) = (x - y[n+1]_q - [n]_q) L_n(x; q) - y[n]_q^2 L_{n-1}(x; q). \quad (14)$$

We derive then the explicit formula for $L_n(x)$:

$$L_n(x; q) = \sum_{k=0}^n (-1)^{n-k} \frac{n!_q}{k!_q} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} y^{n-k} \prod_{j=0}^{k-1} (x - (1 - yq^{-j})[j]_q). \quad (15)$$

Thus

$$\begin{aligned} L_1(x; q) &= x - y, \\ L_2(x; q) &= x^2 - (1 + 2y + qy)x + (1 + q)y^2, \\ L_3(x; q) &= x^3 - (q^2y + 3y + q + 2 + 2qy)x^2 \\ &\quad + (q^3y^2 + yq^2 + q + 2qy + 3q^2y^2 + 1 + 4qy^2 + 2y + 3y^2)x \\ &\quad - (2q^2 + 2q + q^3 + 1)y^3 \end{aligned}$$

A combinatorial interpretation of these q -Laguerres polynomials can be derived from the Simion and Stanton's combinatorial model for octabasic Laguerre polynomials [19]. For a subset A of $[n]$, the *functional digraph* of an injection $f : A \rightarrow [n]$ consists of disjoint paths and cycles. Each path P is of the form $a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_l$, where $f(a_j) = a_{j+1}$ for $0 \leq j < l$, with $f^{-1}(a_0)$ empty, and $a_l \in [n] - A$. We put $\text{last}(P) = a_l$ and if $i = a_k \in P$ we write $\text{ind}(i, P) = k$ for the index of i on the path P . For any path P in the digraph and two integers $i < j$, we put

$$n_P(i, j) = |\{a \in P : i < a < j\}|.$$

For $p \in P$ and two integers $i < j$, we define

$$m_P(p; i, j) = |\{a \in P : i < a < j, \text{ind}(p, P) < \text{ind}(a, P)\}|,$$

that is, the number of points on the path "to the right" of p , whose values are strictly between i and j . And finally, for $i \in A$, we denote by $F(i)$ the "first forward iterate" of f which is smaller than i , i.e.,

$$F(i) = \begin{cases} f^p(i), & \text{where } p = \min\{m \geq 1, f^m(i) < i \text{ if such } m \text{ exists}\}; \\ i, & \text{if } \{m \geq 1, f^m(i) < i\} \text{ is empty.} \end{cases}$$

For instance, suppose that the path $P = 2 \rightarrow 7 \rightarrow 1 \rightarrow 5 \rightarrow 3$ is a connected component of the functional diagram of f . Then $n_P(1, 4) = |\{2, 3\}| = 2$, $m_P(7; 1, 4) = |\{3\}| = 1$, and $F(2) = F(7) = 1$, $F(1) = 1$, and $F(5) = 3$.

For any $k \in [n]$, let $\alpha(k) = w(k) = 0$ if $k \notin A$, otherwise if k is on a cycle or a path P such that $k > \text{last}(P)$, then $\alpha(k) = 1$ and

$$w(k) = F(k) - 1 - \sum_{\text{last}(Q) > k} n_Q(0, F(k));$$

if k is on a path P such that $k < \text{last}(P)$, then $\alpha(k) = 0$ and

$$w(k) = k - 1 - m_P(k; 0, k) - \sum_{\text{last}(Q) > \text{last}(P)} n_Q(0, k),$$

where Q ranges over all paths in the functional digraphs of f . Let

$$w(A, f) = \sum_{k \in A} w(k) \quad \text{and} \quad \alpha(A, f) = \sum_{k \in A} \alpha(k).$$

Example 1. Let $n = 9$, $A = \{2, 9\}$ and $\sigma = (6)(47)(3518)$ (in cycle notation with maximum at last). Then we have $\text{cyc}(\sigma) = 3$ and

$$w(A, \sigma) = (3 - 1 - 1) + (5 - 1 - 1) + (1 - 1) + (4 - 1 - 2) = 5.$$

Theorem 2. *The q -Laguerre polynomials have the following interpretation:*

$$L_n(x; q) = \sum_{A \subset [n], f: A \rightarrow [n]} (-1)^{|A|} x^{n-|A|} y^{\alpha(A,f)} q^{w(A,f)},$$

where f is injective.

Proof. This is the $a = 1$, $s = u = 1$ and $r = t = q$ special case of the quadrabasic Laguerre polynomials [19, p.313]. \square

Remark 1. *It is easy to see that the constant term $L_n(0)$ is equal to*

$$L_n(0) = (-1)^n y^n n!_q.$$

So the restriction of the statistic on permutations is a Mahonian statistic.

4. MOMENTS OF THE q -LAGUERRE POLYNOMIALS

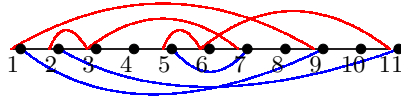
Let \mathcal{S}_n be the set of permutations of $[n] := \{1, 2, \dots, n\}$. For $\sigma \in \mathcal{S}_n$ the *crossing number* of σ is defined by

$$cr(\sigma) = \sum_{i=1}^n \#\{j | j < i \leq \sigma(j) < \sigma(i)\} + \sum_{i=1}^n \#\{j | j > i > \sigma(j) > \sigma(i)\},$$

while the number of *weak excedances* of σ is defined by

$$wex(\sigma) = \#\{i | 1 \leq i \leq n \text{ and } i \leq \sigma(i)\}.$$

We can depict these statistics by associating with each permutation σ of $[n]$ a diagram by drawing an arc $i \rightarrow \sigma(i)$ above (resp. under) the segment $1 \rightarrow 2 \rightarrow \dots \rightarrow n$ if $i \leq \sigma(i)$ (resp. $i > \sigma(i)$). For example, the permutation $\sigma = 9374611581102$ can be depicted as follows:



Let $\mu_n^{(\ell)}(y, q)$ be the enumerating polynomial of permutations in \mathcal{S}_n with respect to weak excedances and crossing numbers:

$$\mu_n^{(\ell)}(y, q) := \sum_{\sigma \in \mathcal{S}_n} y^{wex(\sigma)} q^{cr(\sigma)}.$$

Randrianarivony [17] and Corteel [3] have proved the following continued fraction expansion:

$$E(y, q, t) := \sum_{n \geq 0} \mu_n^{(\ell)}(y, q) t^n = \frac{1}{1 - b_0 t - \frac{\lambda_1 t^2}{1 - b_1 t - \frac{\lambda_2 t^2}{\ddots}}}, \quad (16)$$

where $b_n = y[n+1]_q + [n]_q$ and $\lambda_n = y[n]_q^2$.

We derive then from the classical theory of orthogonal polynomials the following interpretation of the moments of the q -Laguerre polynomials.

Theorem 3. *The n -th moment of the q -Laguerre polynomials is equal to $\mu_n^{(\ell)}(y, q)$. More precisely, let \mathcal{L}_q be the linear functional defined by $\mathcal{L}_q(x^n) = \mu_n^{(\ell)}(y, q)$, then*

$$\mathcal{L}_q(L_{n_1}(x; q)L_{n_2}(x; q)) = y^{n_1}(n_1!_q)^2 \delta_{n_1 n_2}. \quad (17)$$

The first values of the moment sequence are as follows:

$$\begin{aligned} \mu_1^{(\ell)}(y, q) &= y, \\ \mu_2^{(\ell)}(y, q) &= y + y^2, \\ \mu_3^{(\ell)}(y, q) &= y + (3 + q)y^2 + y^3, \\ \mu_4^{(\ell)}(y, q) &= y + (6 + 4q + q^2)y + (6 + 4q + q^2)y^3 + y^4. \end{aligned}$$

Combining the results of Corteel [3], Williams [21, Proposition 4.11] and the classical theory of orthogonal polynomials, one can write the moments of the above q -Laguerre polynomials as a finite double sum (cf. (28)). Here we propose a direct proof of this result. Actually we shall give such a formula for the moments of Al-Salam-Chihara polynomials.

Definition 4. *Define the y -versions of the q -Stirling numbers of the second kind by*

$$X^n = \sum_{k=1}^n S_q(n, k, y) \prod_{j=0}^{k-1} (X - [j]_q(1 - yq^{-j})). \quad (18)$$

The y -versions of q -Stirling numbers of the first kind can be defined by the inverse matrix or equivalently

$$\prod_{j=0}^{n-1} (X - [j]_q(1 - yq^{-j})) = \sum_{k=1}^n s_q(n, k, y) X^k.$$

Remark 2. *We have*

$$S_q(n, k, y)|_{q=1} = S(n, k)(1 - y)^{n-k}, \quad S_q(n, k, 0) = S_q(n, k),$$

where $S(n, k)$ and $S_q(n, k)$ are, respectively, the Stirling numbers of the second kind and their well-known q -analogues, see [11].

Consider the rescaled Al-Salam-Chihara polynomials $P_n(x)$:

$$\begin{aligned} P_n(X) &= Q_n(((q-1)X + 1/\alpha^2 + 1)\alpha/2; \alpha, \beta|q) \\ &= \alpha^{-n} \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q; q)_k} q^k (\alpha\beta q^k; q)_{n-k} (1-q)^k q^{\binom{k}{2}} \alpha^{2k} \\ &\quad \times \prod_{j=0}^{k-1} (X - [j]_q(1 - q^{-j}/\alpha^2)). \end{aligned} \quad (19)$$

Theorem 1. *The moments of the rescaled Al-Salam-Chihara polynomials $P_n(X)$ are*

$$\mu_n(\alpha, \beta) = \sum_{k=1}^n S_q(n, k, 1/\alpha^2) (\alpha\beta; q)_k q^{-\binom{k}{2}} (1-q)^{-k} \alpha^{-2k}.$$

Proof. Let $L : X^n \mapsto \mu_n(\alpha, \beta)$ be the linear functional. We check that these moments do satisfy $L(P_n(X)) = 0$ for $n > 0$. Let a_k be the coefficients in front of the product in (19), then we have, using y -Stirling orthogonality,

$$\begin{aligned} L(P_n(X)) &= \sum_{k=0}^n a_k \sum_{j=1}^k s_q(k, j, 1/\alpha^2) \sum_{t=1}^j S_q(j, t, 1/\alpha^2) (\alpha\beta; q)_t q^{-\binom{t}{2}} (1-q)^{-t} \alpha^{-2t} \\ &= \sum_{k=0}^n a_k (\alpha\beta; q)_k q^{-\binom{k}{2}} (1-q)^{-k} \alpha^{-2k} \\ &= \alpha^{-n} (\alpha\beta; q)_n \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q; q)_k} q^k = 0. \end{aligned}$$

Note that the last equality follows by applying the q -binomial formula. \square

Theorem 2. *The generating function for the moments $\mu_n(\alpha, \beta)$ is*

$$\sum_{n=0}^{\infty} \mu_n(\alpha, \beta) t^n = \sum_{k=0}^{\infty} \frac{(\alpha\beta; q)_k q^{-\binom{k}{2}} (1-q)^{-k} \alpha^{-2k} t^k}{\prod_{i=1}^k (1 - [i]_q t (1 - q^{-i}/\alpha^2))}. \quad (20)$$

Proof. By definition (18) we have

$$S_q(n, k, y) = S_q(n-1, k-1, y) + [k]_q (1 - yq^{-k}) S_q(n-1, k, y).$$

It follows that (18) is equivalent to

$$\sum_{n \geq k} S_q(n, k, y) t^n = \frac{t^k}{\prod_{i=1}^k (1 - [i]_q t (1 - q^{-i}y))}, \quad (21)$$

which yields immediately (20) in view of Theorem 1. \square

The moment of q -Charlier polynomials corresponds to the $\beta = 0$, $\alpha = -1/\sqrt{a(1-q)}$ case, while that of q -Laguerre polynomials corresponds to the $\alpha = 1/\sqrt{y}$, $\alpha\beta = q$ case. Therefore,

$$\sum_{n=0}^{\infty} \mu_n^{(c)}(a, q) t^n = \sum_{k=0}^{\infty} \frac{a(qt)^k}{\prod_{i=1}^k (q^i - q^i [i]_q t + a(1-q)[i]_q t)}, \quad (22)$$

$$\sum_{n=0}^{\infty} \mu_n^{(\ell)}(y, q) t^n = \sum_{k=0}^{\infty} \frac{k!_q (qty)^k}{\prod_{i=1}^k (q^i - q^i [i]_q t + [i]_q qty)}. \quad (23)$$

Theorem 3. *Let $p = 1/q$. We have*

$$\sum_{k=0}^{\infty} \frac{(\alpha\beta; q)_k q^{-\binom{k}{2}} (1-q)^{-k} \alpha^{-2k} t^k}{\prod_{i=1}^k (1 - [i]_q t (1 - q^{-i}/\alpha^2))} = \sum_{i \geq 0} \frac{c_i(\alpha, \beta)}{1 - [i]_q t (1 - q^{-i}/\alpha^2)}, \quad (24)$$

where

$$c_i(\alpha, \beta) = \frac{(\alpha\beta; q)_i}{(q; q)_i} \frac{q^{i-i^2} \alpha^{-2i}}{(q^{1-2i}/\alpha^2; q)_i} \frac{(p^{1+i} \alpha \beta / \alpha^2; p)_{\infty}}{(p^{1+2i} / \alpha^2; p)_{\infty}}.$$

Proof. Note the following partial fraction decomposition formula:

$$\frac{t^k}{(1-a_1t)(1-a_2t)\dots(1-a_kt)} = \frac{(-1)^k}{a_1 \cdots a_k} + \sum_{i=1}^k \frac{a_i^{-1} \prod_{j=1, j \neq i}^k (a_i - a_j)^{-1}}{1 - a_i t}.$$

Therefore

$$\frac{t^k}{\prod_{i=1}^k (1 - [i]_q t (1 - q^{-i}/\alpha^2))} = \sum_{i=0}^k \frac{\gamma_k(i)}{1 - [i]_q t (1 - q^{-i}/\alpha^2)}, \quad (25)$$

where

$$\gamma_k(i) = \frac{1}{k!_q} \begin{bmatrix} k \\ i \end{bmatrix}_q \frac{\alpha^{2(k-i)} q^{\binom{k}{2} + k - i^2}}{(q^{1-2i}/\alpha^2; q)_i (q^{1+2i}\alpha^2; q)_{k-i}} \quad (0 \leq i \leq k).$$

Substituting this in (24) yields

$$\begin{aligned} c_i(\alpha, \beta) &= \sum_{k \geq i} \frac{(\alpha\beta; q)_k}{(q; q)_k} \begin{bmatrix} k \\ i \end{bmatrix}_q \frac{q^{k-i^2} \alpha^{-2i}}{(q^{1-2i}/\alpha^2; q)_i (q^{1+2i}\alpha^2; q)_{k-i}} \\ &= \frac{(\alpha\beta; q)_i}{(q; q)_i} \frac{q^{i-i^2} \alpha^{-2i}}{(q^{1-2i}/\alpha^2; q)_i} \sum_{k \geq 0} \frac{(\alpha\beta q^i; q)_k}{(q; q)_k} \frac{q^k}{(q^{1+2i}\alpha^2; q)_k}. \end{aligned}$$

The theorem follows then by applying the ${}_1\Phi_1$ summation formula (see [10, II.5]). \square

By partial fraction decomposition (see [21, Theorem 4.12]), we get

$$\sum_{n=0}^{\infty} \mu_n^{(c)}(a, q) t^n = \sum_{i \geq 0} \frac{a^i (1 - a(1-q)p^{2i}) / (a(1-q)p^i; p)_{\infty}}{i!_q q^{i^2} (q^i - q^i [i]_q t + a[i]_q t (1-q))}, \quad (26)$$

$$\sum_{n=0}^{\infty} \mu_n^{(\ell)}(y, q) t^n = \sum_{i \geq 0} \frac{y^i (q^{2i} - y)}{q^{i^2} (q^i - q^i [i]_q t + [i]_q t y)}. \quad (27)$$

Note that (27) yields the following polynomial formula in y for $\mu_n^{(\ell)}(y, q)$:

$$\mu_n^{(\ell)}(y, q) = \sum_{k=1}^n \sum_{i=0}^{k-1} (-1)^i [k-i]_q^n q^{k(i-k)} \left(\binom{n}{i} q^{k-i} + \binom{n}{i-1} \right) y^k, \quad (28)$$

while (26) does not yield such a polynomial formula in a for $\mu_n^{(c)}(a, q)$.

On the other hand, it follows from (25) and (21) that

$$S_q(n, k, y) = \frac{q^{-\binom{k}{2}}}{k!_q} \sum_{i=1}^k \begin{bmatrix} k \\ i \end{bmatrix}_q y^{i-k} q^{k^2-i^2} \frac{([i]_q (1 - q^{-i}y))^n}{(q^{1-2i}y; q)_i (q^{1+2i}/y; q)_{k-i}}. \quad (29)$$

Using Theorem 1 and the above explicit formula for q -Stirling numbers we can also write the moments $\mu_n(\alpha, \beta)$ as a double sum.

5. LINEARIZATION COEFFICIENTS OF THE q -LAGUERRE POLYNOMIALS

The following is our main result of this section.

Theorem 5. *The linearization coefficients of the q -Laguerre polynomials are*

$$\mathcal{L}_q(L_{n_1}(x; q) \dots L_{n_k}(x; q)) = \sum_{\sigma \in \mathcal{D}(n_1, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}. \quad (30)$$

A proof à la Viennot (cf. [12, 15]) of (30) would use the combinatorial interpretations for the moments and q -Laguerre polynomials to rewrite the left-hand side of (30) and then construct an adequate *killing involution* on the resulting set. For the time being we do not have such a proof to offer, instead we provide an inductive proof.

We first show that the above result is true for $(n_1, \dots, n_k) = (1, \dots, 1)$.

Lemma 6. *Let $d_n(y, q) = \sum_{\sigma \in \mathcal{D}_n} y^{wex(\sigma)} q^{cr(\sigma)}$. Then $\mathcal{L}_q((x - y)^n) = d_n(y, q)$.*

Proof. Note that

$$\mathcal{L}_q((x - y)^n) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} y^{n-k} \mu_k^{(\ell)}(y, q).$$

By binomial inversion, it suffices to prove that

$$\mu_n^{(\ell)}(y, q) = \sum_{k=0}^n \binom{n}{k} y^k d_{n-k}(y, q).$$

But the latter identity is obvious. □

The invariance of $\sum_{\sigma \in \mathcal{D}(n_1, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}$ by permutating the n'_i 's is a direct consequence of Theorem 5, but for our proof we need to first establish this property.

Theorem 7. *For any permutation $\gamma \in \mathcal{S}_k$ we have*

$$\sum_{\sigma \in \mathcal{D}(n_1, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)} = \sum_{\sigma \in \mathcal{D}(n_{\gamma(1)}, n_{\gamma(2)}, \dots, n_{\gamma(k)})} y^{wex(\sigma)} q^{cr(\sigma)}.$$

Since the two cyclic permutations $(1, 2)$ and $(1, 2, 3, \dots, k)$ generate the symmetric group \mathcal{S}_k , Theorem 7 is a corollary of the following two lemmas (proved in the next two sections).

Lemma 8.

$$\sum_{\sigma \in \mathcal{D}(n_1, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)} = \sum_{\sigma \in \mathcal{D}(n_2, n_3, \dots, n_k, n_1)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

Lemma 9.

$$\sum_{\sigma \in \mathcal{D}(n_1, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)} = \sum_{\sigma \in \mathcal{D}(n_2, n_1, n_3, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

Proof of Theorem 4. Writing (14) as

$$(x - y)L_n(x) = L_{n+1}(x) + (yq + 1)[n]_q L_n(x) + y[n]_q^2 L_{n-1}(x),$$

we derive that

$$\sum_{\sigma \in \mathcal{D}(1, n, n_2, \dots, n_k)} w(\pi) = \sum_{\sigma \in \mathcal{D}(n+1, n_2, \dots, n_k)} w(\pi) + (yq + 1)[n]_q \sum_{\sigma \in \mathcal{D}(n, n_2, \dots, n_k)} w(\pi) + y[n]_q^2 \sum_{\sigma \in \mathcal{D}(n-1, n_2, \dots, n_k)} w(\pi), \quad (31)$$

where $w(\pi) = y^{wex(\sigma)} q^{cr(\sigma)}$. In view of Lemma 6 it suffices to prove (31).

We distinguish four cases for permutations $\pi \in \mathcal{D}(1, n, n_2, \dots, n_k)$.

a) $\pi(1), \pi^{-1}(1) \in \{2, \dots, n+1\}$. Let $\pi(1) = i$ and $\pi(j) = 1$ with $i, j \in \{2, \dots, n+1\}$.

Then we define the mapping $\pi \rightarrow \pi' \in \mathcal{D}(n-1, n_2, \dots, n_k)$ by deleting 1 and j and adding the edge $\pi^{-1}(j) \rightarrow i$ if $i \neq j$. Clearly

$$w(\pi) = yq^{(i-1)+(j-1)-2} w(\pi').$$

Summing over all $i, j \in \{2, \dots, n+1\}$ yields the generating function:

$$y[n]_q^2 \sum_{\sigma \in \mathcal{D}(n-1, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

b) $\pi(1) \in \{2, \dots, n+1\}$ and $\pi^{-1}(1) > n+1$. We define the mapping $\pi \rightarrow \pi' \in \mathcal{D}(n, n_2, \dots, n_k)$ by deleting $i := \pi(1)$ and replace the two edges $1 \rightarrow \pi(1) \rightarrow \pi^2(1)$ by $1 \rightarrow \pi^2(1)$. Clearly $w(\pi) = yq^{i-1} w(\pi')$. Summing over all $i = 2, \dots, n+1$ yields the generating function:

$$qy[n]_q \sum_{\sigma \in \mathcal{D}(n, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

c) $\pi^{-1}(1) \in \{2, \dots, n+1\}$ and $\pi(1) > n+1$. We define the mapping $\pi \rightarrow \pi' \in \mathcal{D}(n, n_2, \dots, n_k)$ by deleting $i := \pi^{-1}(1)$ and replace the two edges $1 \leftarrow \pi^{-1}(1) \leftarrow \pi^{-2}(1)$ by $1 \leftarrow \pi^{-2}(1)$. Clearly $w(\pi) = q^{i-2} w(\pi')$. Summing over all $i = 2, \dots, n+1$ yields the generating function:

$$[n]_q \sum_{\sigma \in \mathcal{D}(n, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

d) $\pi(1) > n+1$ and $\pi^{-1}(1) > n+1$. Clearly we can consider π as a permutation in $\mathcal{D}(n+1, n_2, \dots, n_k)$. The generating function is

$$\sum_{\sigma \in \mathcal{D}(n+1, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

Summing up we obtain (31). \square

When $k = 2$ Theorem 4 reduces to the orthogonality of the q -Laguerre polynomials (17). When $k = 3$ we can derive the following explicit formula from Theorem 1.

Theorem 10. *We have*

$$\begin{aligned} \mathcal{L}_q(L_{n_1}(x; q) L_{n_2}(x; q) L_{n_3}(x; q)) &= \sum_s \frac{n_1!_q n_2!_q n_3!_q s!_q y^s}{(n_1 + n_2 + n_3 - 2s)!_q (s - n_3)!_q (s - n_2)!_q (s - n_1)!_q} \\ &\quad \times \sum_k \left[\begin{matrix} n_1 + n_2 + n_3 - 2s \\ k \end{matrix} \right]_q y^k q^{\binom{k+1}{2} + (n_1 + n_2 + n_3 - 2s - k)}. \end{aligned}$$

Proof. By Theorem 1 with $a = \frac{1}{\sqrt{y}}$ and $b = \sqrt{y}q$ we have

$$\begin{aligned} & \mathcal{L}_q(L_{n_1}(x; q)L_{n_2}(x; q)L_{n_3}(x; q)) \\ &= \mathcal{L}_q(L_{n_3}(x; q)^2) \left(\frac{\sqrt{y}}{q-1} \right)^{n_1+n_2-n_3} C_{n_1, n_2}^{n_3}(a, b; q) \\ &= \sum_{m_2, m_3} \frac{n_1!_q n_2!_q n_3!_q (n_1 + m_3)!_q y^{n_2+n_3-m_2-m_3} q^{\binom{m_2}{2} + \binom{n_3+n_2-n_1-m_2-2m_3+1}{2}}}{(n_3 + n_2 - n_1 - m_2 - 2m_3)!_q m_2!_q (m_3 + n_1 - n_3)!_q (m_3 + n_1 - n_2)!_q m_3!_q}. \end{aligned}$$

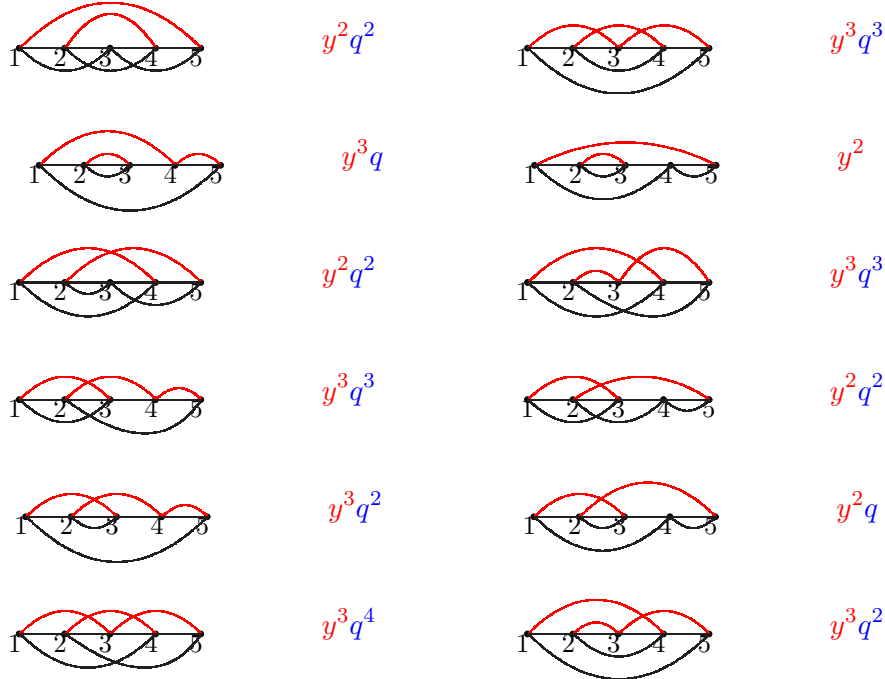
Substituting $s = n_1 + m_3$ and $k = n_3 + n_2 - n_1 - m_2 - 2m_3$ in the last sum yields the desired formula. \square

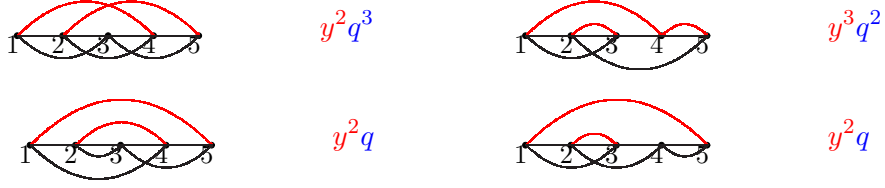
Remark 3. It would be interesting to give a combinatorial proof of the above result as in [12, 15]. When $q = 1$ such a proof was given in [23].

We end this section with an example. If $\mathbf{n} = (2, 2, 1)$, by Theorem 8 we have

$$\begin{aligned} \mathcal{L}_q(L_2(x; q)L_2(x; q)L_1(x; q)) &= \sum_s \frac{2!_q 2!_q 1!_q s!_q y^s}{(5-2s)!_q (s-1)!_q (s-2)!_q (s-2)!_q} \\ &\quad \times \sum_{k \geq 0} \begin{bmatrix} 5-2s \\ k \end{bmatrix}_q y^k q^{\binom{k+1}{2} + \binom{5-2s-k}{2}} \\ &= (1+q)^3 (1+qy)y^2. \end{aligned} \tag{32}$$

On the other hand, the sixteen derangements, depicted by their diagrams and the corresponding weights are tabulated as follows:





Summing up we get $\sum_{\sigma \in D(2,2,1)} y^{wex\sigma} q^{cr\sigma} = y^2(1 + qy)(1 + q)^3$, which coincides with (32).

6. PROOF OF LEMMA 8

For each fixed $k \in [n]$ define the two subsets of \mathcal{S}_n :

$$\begin{aligned} {}^k\mathcal{S}_n &= \{\sigma \in \mathcal{S}_n \mid \sigma(i) > k \text{ for } 1 \leq i \leq k\}, \\ \mathcal{S}_n^k &= \{\sigma \in \mathcal{S}_n \mid \sigma(n+1-i) < n+1-k \text{ for } 1 \leq i \leq k\}. \end{aligned}$$

We first construct a simple bijection $\Phi_k : {}^k\mathcal{S}_n \rightarrow \mathcal{S}_n^k$. Let $\sigma \in {}^k\mathcal{S}_n$. For $1 \leq i \leq n$ we define $\sigma'(i) := \Phi_k(\sigma)(i)$ as follows:

$$\sigma'(i) = \begin{cases} \sigma(i+k) - k, & \text{if } 1 \leq i \leq n-k \text{ and } \sigma(i+k) > k; \\ \sigma(i+k) + n - k, & \text{if } 1 \leq i \leq n-k \text{ and } \sigma(i+k) \leq k; \\ \sigma(i+k-n) - k, & \text{if } n-k+1 \leq i \leq n. \end{cases}$$

We can illustrate the map by the diagrams of permutations.

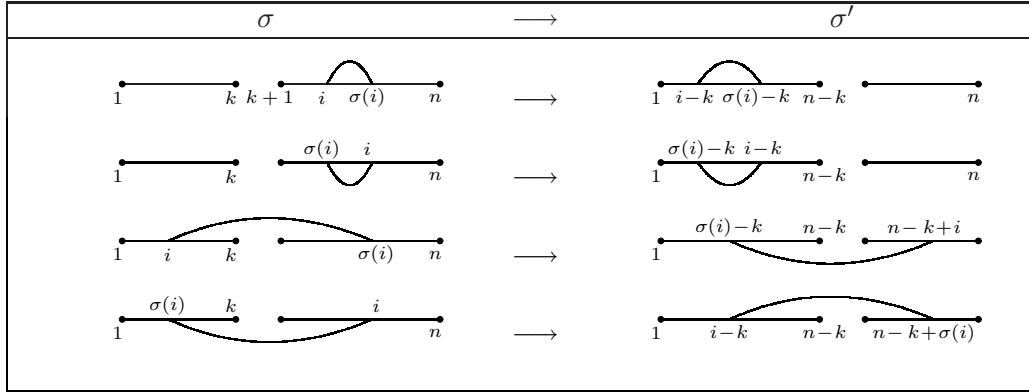
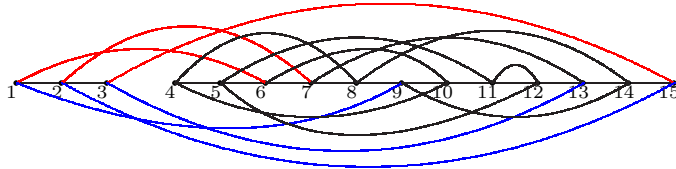
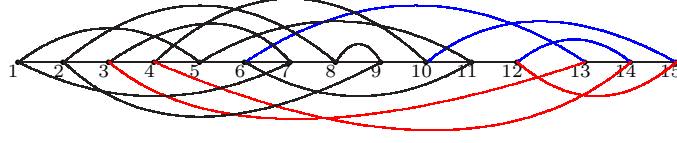


TABLE 1. The mapping $\Phi_k : \sigma \rightarrow \sigma'$.

For example, consider the permutation $\sigma \in {}^3\mathcal{S}_{15}$, whose diagram is given below.



Then the diagram of $\Phi_3(\sigma)$ is given by



The main properties of Φ_k are summarized in the following result.

Lemma 11. *For each positive integer $k \in [n]$, the map $\Phi_k : {}^k\mathcal{S}_n \rightarrow \mathcal{S}_n^k$ is a bijection such that for any $\sigma \in {}^k\mathcal{S}_n$ there holds*

$$(wex, cr)\Phi_k(\sigma) = (wex, cr)\sigma. \quad (33)$$

Now, Lemma 8 is an immediate consequence of Lemma 11. Let $n = n_1 + n_2 + \dots + n_k$. Then $\mathcal{D}(n_1, n_2, \dots, n_k) \subseteq {}^{n_1}\mathcal{S}_n$. By definition of Φ_{n_1} , for any $\sigma \in {}^{n_1}\mathcal{S}_n$ and $i \in [n - n_1]$ satisfying $\sigma(i + n_1) > n_1$, we have $i - \Phi_{n_1}(\sigma)(i) = i + n_1 - \sigma(i + n_1)$, so $\Phi_{n_1}(\mathcal{D}(n_1, n_2, \dots, n_k)) \subseteq \mathcal{D}(n_2, n_3, \dots, n_k, n_1)$. Since the cardinality of $\mathcal{D}(n_1, n_2, \dots, n_k)$ is invariant by permutations of the n_i 's and Φ_{n_1} is bijective, we have $\Phi_{n_1}(\mathcal{D}(n_1, n_2, \dots, n_k)) = \mathcal{D}(n_2, n_3, \dots, n_k, n_1)$. The result follows then by applying (33).

i	$L_i(\sigma)$	$R_i(\sigma')$
1		
2		
3		

TABLE 2. Forms of crossings in $L_i(\sigma)$ and $R_i(\sigma')$.

Proof of Lemma 11 It is easy to see that Φ_k is a bijection. Let $\sigma \in {}^k\mathcal{S}_n$ and $\sigma' = \Phi_k(\sigma)$. The equality $wex(\sigma') = wex(\sigma)$ follows directly from the definition of Φ_k . It then remains to prove

that $cr(\sigma') = cr(\sigma)$. We first decompose the crossings of σ and σ' into three subsets. Set

$$L_1(\sigma) = \{(i, j) \mid k < i < j \leq \sigma(i) < \sigma(j) \text{ or } i > j > \sigma(i) > \sigma(j) > k\},$$

$$L_2(\sigma) = \{(i, j) \mid i < j \leq k < \sigma(i) < \sigma(j) \text{ or } i > j > k \geq \sigma(i) > \sigma(j)\},$$

$$L_3(\sigma) = \{(i, j) \mid i \leq k < j \leq \sigma(i) < \sigma(j) \text{ or } i > j > \sigma(i) > k \geq \sigma(j)\},$$

and

$$R_1(\sigma') = \{(i, j) \mid i < j \leq \sigma'(i) < \sigma'(j) \leq n - k \text{ or } n - k \geq i > j > \sigma'(i) > \sigma'(j)\},$$

$$R_2(\sigma') = \{(i, j) \mid i < j \leq n - k < \sigma'(i) < \sigma'(j) \text{ or } i > j > n - k \geq \sigma'(i) > \sigma'(j)\},$$

$$R_3(\sigma') = \{(i, j) \mid i < j \leq \sigma'(i) \leq n - k < \sigma'(j) \text{ or } i > n - k \geq j > \sigma(i) > \sigma(j)\}.$$

The "forms" of the crossings in the L_i 's and R_i 's is given in Table 2. Clearly, we have $cr(\sigma) = \sum_{i=1}^3 |L_i(\sigma)|$ and $cr(\sigma') = \sum_{i=1}^3 |R_i(\sigma')|$ since $\sigma \in {}^k\mathcal{S}_n$ and $\sigma' \in \mathcal{S}_n^k$.

σ	\rightarrow	σ'
	\rightarrow	
	\rightarrow	
	\rightarrow	
	\rightarrow	

TABLE 3. Effects of the mapping Φ_k on the crossings of σ and σ' .

By the definition of Φ_k , it is readily seen (see Row 1 in Table 3) that $(i, j) \in L_1(\sigma)$ if and only if $(i - k, j - k) \in R_1(\sigma')$, and thus $|L_1(\sigma)| = |R_1(\sigma')|$. Similarly, we have (see Row 2 in Table 3) that $|L_2(\sigma)| = |R_2(\sigma')|$. It then remains to prove that $|L_3(\sigma)| = |R_3(\sigma')|$. Let

$$L_4(\sigma) = \{(i, j) \mid \sigma(i) \leq k < j < i \leq \sigma(j) \text{ or } i \leq k < \sigma(j) < \sigma(i) < j\}.$$

Then it is not difficult to show (see Row 4 of Table 3) that $|R_3(\sigma')| = |L_4(\sigma)|$. It then suffices to prove that $|L_3(\sigma)| = |L_4(\sigma)|$.

Suppose $\sigma([1, k]) = \{i_1, i_2, \dots, i_k\}$ and $\sigma^{-1}([1, k]) = \{j_1, j_2, \dots, j_k\}$. Then by definition of $L_3(\sigma)$ and $L_4(\sigma)$ we have

$$|L_3(\sigma)| = \sum_{s=1}^k |\{\ell \mid k < \ell \leq i_s < \sigma(\ell)\}| + |\{\ell \mid \ell > j_s > \sigma(\ell) > k\}|, \quad (34)$$

$$|L_4(\sigma)| = \sum_{s=1}^k |\{\ell \mid \ell > i_s > \sigma(\ell) > k\}| + |\{\ell \mid k < \ell < j_s \leq \sigma(\ell)\}|. \quad (35)$$

For any integer $i \in [n]$ set $A_i(\sigma) = \{j \mid j < i < \sigma(j)\}$. Note that it is easily seen that

$$|A_i(\sigma)| = |\{j \mid j < i < \sigma(j)\}| = |\{j \mid j > i > \sigma(j)\}| = |A_i(\sigma^{-1})|. \quad (36)$$

Let $s \in [k]$. By elementary manipulations we get

$$\begin{aligned} |\{\ell \mid k < \ell \leq i_s < \sigma(\ell)\}| &= |\{\ell \mid \ell \leq i_s < \sigma(\ell)\}| - |\{\ell \mid \ell \leq k < i_s < \sigma(\ell)\}| \\ &= |A_{i_s}(\sigma)| + \chi(i_s < \sigma(i_s)) - |\{\ell \mid \ell \leq k < i_s < \sigma(\ell)\}| \\ &= |A_{i_s}(\sigma)| + \chi(i_s < \sigma(i_s)) - |\{t \mid i_t > i_s\}|. \end{aligned} \quad (37)$$

By a similar reasoning, we obtain the following identities:

$$|\{\ell \mid \ell > j_s > \sigma(\ell) > k\}| = |A_{j_s}(\sigma^{-1})| - |\{t \mid j_t > j_s\}|. \quad (38)$$

$$|\{\ell \mid \ell > i_s > \sigma(\ell) > k\}| = |A_{i_s}(\sigma^{-1})| - |\{t \mid j_t > i_s\}| \quad (39)$$

$$|\{\ell \mid k \leq \ell < j_s \leq \sigma(\ell)\}| = |A_{j_s}(\sigma)| + \chi(k < \sigma^{-1}(j_s) < j_s) - |\{t \mid i_t > j_s\}|. \quad (40)$$

Inserting (37) and (38) in (34) and using (36), we get

$$|L_3(\sigma)| = \sum_{s=1}^k |A_{i_s}(\sigma)| + |A_{j_s}(\sigma)| + \chi(i_s < \sigma(i_s)) - |\{t \mid i_t > i_s\}| - |\{t \mid j_t > j_s\}|. \quad (41)$$

Similarly, inserting (39) and (40) in (35) and using (36), we get

$$|L_4(\sigma)| = \sum_{s=1}^k |A_{i_s}(\sigma)| + |A_{j_s}(\sigma)| + \chi(k < \sigma^{-1}(j_s) < j_s) - |\{t \mid j_t > i_s\}| - |\{t \mid i_t > j_s\}|. \quad (42)$$

Since the i_t 's and j_t 's are distinct we have $\sum_{s=1}^k |\{t \mid i_t > i_s\}| = \sum_{s=1}^k |\{t \mid j_t > j_s\}| = \binom{k}{2}$ and thus

$$\sum_{s=1}^k |\{t \mid i_t > i_s\}| + |\{t \mid j_t > j_s\}| = k(k-1). \quad (43)$$

On the other hand,

$$\begin{aligned}
\sum_{s=1}^k |\{t \mid j_t > i_s\}| + |\{t \mid i_t > j_s\}| &= \sum_{s=1}^k |\{t \mid i_t \neq j_s\}| \\
&= k^2 - \sum_{s=1}^k \chi(j_s \in \{i_1, i_2, \dots, i_k\}) \\
&= k^2 - \sum_{s=1}^k \chi(\sigma^{-1}(j_s) \leq k). \tag{44}
\end{aligned}$$

Also, it follows from the definition of the j_t 's that for any $s \in [k]$, we have $j_s > k$ and $\sigma^{-1}(j_s) \neq j_s$, and thus

$$\chi(k < \sigma^{-1}(j_s) < j_s) + \chi(\sigma^{-1}(j_s) > j_s) = \chi(\sigma^{-1}(j_s) > k). \tag{45}$$

Inserting (53), (54) and (45) in (41) and (42) lead to $|L_3(\sigma)| = |L_4(\sigma)|$ as desired.

7. PROOF OF LEMMA 9

For any two integers n_1, n_2 satisfying $N_2 := n_1 + n_2 \leq n$ we denote by $\mathcal{S}_n^{(n_1, n_2)}$ the set of permutations σ in \mathcal{S}_n such that

$$(i, \sigma(i)) \notin [1, n_1]^2 \cup [n_1 + 1, N_2]^2.$$

In other words, any two integers in $[1, n_1]$ or $[n_1 + 1, N_2]$ are not connected by an arc in its graph.

We now construct a map $\Gamma^{(n_1, n_2)} : \sigma \mapsto \sigma'$ from $\mathcal{S}_n^{(n_1, n_2)}$ to $\mathcal{S}_n^{(n_2, n_1)}$ as follows. For $i = 1, \dots, n$,

(1) If $i > N_2$ and $\sigma(i) > N_2$, set $\sigma'(i) = \sigma(i)$.

(2) Suppose

$$\begin{aligned}
\{(i, \sigma(i)) \mid i < \sigma(i) \leq N_2\} &= \{(i_1, N_2 + 1 - j_1), (i_2, N_2 + 1 - j_2), \dots, (i_p, N_2 + 1 - j_p)\} \\
\{(\sigma(i), i) \mid \sigma(i) < i \leq N_2\} &= \{(k_1, N_2 + 1 - \ell_1), (k_2, N_2 + 1 - \ell_2), \dots, (k_q, N_2 + 1 - \ell_q)\}.
\end{aligned}$$

Then set $\sigma'(j_s) = N_2 + 1 - i_s$ and $\sigma'(N_2 + 1 - k_t) = \ell_t$ for any $s \in [p]$ and $t \in [q]$.

(3) Let

$$C = \{i \in [1, N_2] \mid \sigma(i) > N_2\} \quad \text{and} \quad D = \{i \in [1, N_2] \mid \sigma^{-1}(i) > N_2\}.$$

It is easy to see that $|C| = |D|$. Suppose $C = \{c_1, c_2, \dots, c_u\}_{<}$, $D = \{d_1, d_2, \dots, d_u\}_{<}$, $\sigma(C) = \{r_1, r_2, \dots, r_u\}_{<}$ and $\sigma^{-1}(D) = \{s_1, s_2, \dots, s_u\}_{<}$. Let $\alpha, \beta \in \mathcal{S}_u$ be the (unique) permutations satisfying $\sigma(c_i) = r_{\alpha(i)}$ and $\sigma^{-1}(d_i) = s_{\beta(i)}$ for each $1 \leq i \leq u$. Let

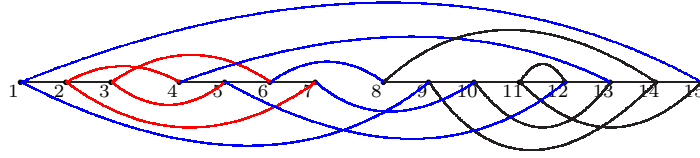
$$\begin{aligned}
E &= [1, N_2] \setminus \{j_1, \dots, j_p, N_2 + 1 - k_1, \dots, N_2 + 1 - k_q\} \\
F &= [1, N_2] \setminus \{N_2 + 1 - i_1, \dots, N_2 + 1 - i_p, \ell_1, \dots, \ell_q\}.
\end{aligned}$$

Clearly, we have $|E| = |C|$ and $|F| = |D|$. Suppose $E = \{e_1, \dots, e_u\}_{<}$ and $F = \{f_1, \dots, f_u\}_{<}$. Then set $\sigma'(e_i) = r_{\alpha(i)}$ and $\sigma'(s_i) = f_{\beta(i)}$ for each $1 \leq i \leq u$.

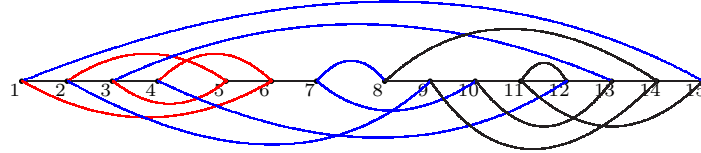
We can illustrate the map through the diagrams of permutations. See Table 4.

For example, if we consider the permutation in $\mathcal{S}_{15}^{(3,4)}$ whose diagram is given by

σ		\longrightarrow	σ'	
$1 \xrightarrow{\quad} N_2$	$i \xrightarrow{\quad} \sigma(i) \xrightarrow{\quad} n$	\longrightarrow	$1 \xrightarrow{\quad} N_2$	$i \xrightarrow{\quad} \sigma(i) \xrightarrow{\quad} n$
$1 \xrightarrow{\quad} N_2$	$\sigma(i) \xrightarrow{\quad} i \xrightarrow{\quad} n$	\longrightarrow	$1 \xrightarrow{\quad} N_2$	$\sigma(i) \xrightarrow{\quad} i \xrightarrow{\quad} n$
$1 \xrightarrow{i_t} n_1 \xrightarrow{N_2+1-j_t} N_2$	$\xrightarrow{\quad} n$	\longrightarrow	$1 \xrightarrow{j_t} n_2 \xrightarrow{N_2+1-i_t} N_2$	$\xrightarrow{\quad} n$
$1 \xrightarrow{k_t} n_1 \xrightarrow{N_2+1-\ell_t} N_2$	$\xrightarrow{\quad} n$	\longrightarrow	$1 \xrightarrow{\ell_t} n_2 \xrightarrow{N_2+1-k_t} N_2$	$\xrightarrow{\quad} n$
$1 \xrightarrow{c_j} N_2 \xrightarrow{r_{\alpha(j)}} n$	\longrightarrow	\longrightarrow	$1 \xrightarrow{e_j} N_2 \xrightarrow{r_{\alpha(j)}} n$	
$1 \xrightarrow{e_j} N_2 \xrightarrow{s_{\beta(j)}} n$	\longrightarrow	\longrightarrow	$1 \xrightarrow{f_j} N_2 \xrightarrow{s_{\beta(j)}} n$	

TABLE 4. The mapping $\Gamma^{(n_1, n_2)} : \sigma \mapsto \sigma'$ 

then the diagram of $\Gamma^{(n_1, n_2)}(\sigma)$ is given by



It is not hard to check that $\Gamma^{(n_1, n_2)}$ is well defined from $\mathcal{S}_n^{(n_1, n_2)}$ to $\mathcal{S}_n^{(n_2, n_1)}$. Since each step of the construction of $\Gamma^{(n_1, n_2)}$ is reversible, the map $\Gamma^{(n_1, n_2)}$ is bijective. Actually we can prove, the details are left to the reader, that $(\Gamma^{(n_1, n_2)})^{-1} = \Gamma^{(n_2, n_1)}$.

Lemma 12. *For each positive integers n_1, n_2, n , with $N_2 \leq n$, the map $\Gamma^{(n_1, n_2)}$ is a bijection from $\mathcal{S}_n^{(n_1, n_2)}$ to $\mathcal{S}_n^{(n_2, n_1)}$ such that for each $\sigma \in \mathcal{S}_n^{(n_1, n_2)}$, we have*

$$(wex, cr)\Gamma^{(n_1, n_2)}(\sigma) = (wex, cr)\sigma. \quad (46)$$

As an immediate consequence, we obtain Lemma 9. Let $n = n_1 + n_2 + \dots + n_k$. Then $\mathcal{D}(n_1, n_2, \dots, n_k) \subseteq \mathcal{S}_n^{(n_1, n_2)}$. By definition of $\Gamma^{(n_1, n_2)}$, for any $\sigma \in \mathcal{S}_n^{(n_1, n_2)}$ and $i > N_2$ satisfying

$\sigma(i) > N_2$, we have

$$i - \Gamma^{(n_1, n_2)}(\sigma)(i) = i - \sigma(i),$$

so $\Gamma^{(n_1, n_2)}(\mathcal{D}(n_1, n_2, \dots, n_k)) \subseteq \mathcal{D}(n_2, n_3, \dots, n_k, n_1)$. Since the cardinality of $\mathcal{D}(n_1, n_2, \dots, n_k)$ doesn't depend of the order of the n_i 's and $\Gamma^{(n_1, n_2)}$ is a bijection, we have

$$\Gamma^{(n_1, n_2)}(\mathcal{D}(n_1, n_2, \dots, n_k)) = \mathcal{D}(n_2, n_3, \dots, n_k, n_1).$$

Lemma 9 then follows from (46).

i	$G_i^{(n_1, n_2)}(\gamma)$	$G_i^{(n_2, n_1)}(\gamma)$
1		
2		
3		
4		
5		

TABLE 5. Forms of the crossings in $G_i^{(n_1, n_2)}(\gamma)$ and $G_i^{(n_2, n_1)}(\gamma)$.

Proof of Lemma 12 It was shown above that $\Gamma^{(n_1, n_2)}$ is bijective. Let $\sigma \in \mathcal{S}_n^{(n_1, n_2)}$ and $\sigma' := \Gamma^{(n_1, n_2)}(\sigma)$. The equality $wex(\sigma') = wex(\sigma)$ is an immediate consequence of the definition

of $\Gamma^{(n_1, n_2)}$. It then remains to prove that $cr(\sigma') = cr(\sigma)$. The idea is the same than for the proof of Lemma 8. We first decompose the number of crossings of σ and σ' . For each permutation $\gamma \in \mathcal{S}_n$, set

$$\begin{aligned} G_1^{(n_1, n_2)}(\gamma) &= \{(i, j) \mid N_2 < i < j \leq \gamma(i) < \gamma(j) \text{ or } i > j > \gamma(i) > \gamma(j) > N_2\}, \\ G_2^{(n_1, n_2)}(\gamma) &= \{(i, j) \mid i < j < \gamma(i) < \gamma(j) \leq N_2 \text{ or } N_2 \geq i > j > \gamma(i) > \gamma(j)\}, \\ G_3^{(n_1, n_2)}(\gamma) &= \{(i, j) \mid i < j \leq N_2 < \gamma(i) < \gamma(j) \text{ or } i > j > N_2 \geq \gamma(i) > \gamma(j)\}, \\ G_4^{(n_1, n_2)}(\gamma) &= \{(i, j) \mid i \leq N_2 < j \leq \gamma(i) < \gamma(j) \text{ or } i > j > \gamma(i) > N_2 \geq \gamma(j)\}, \\ G_5^{(n_1, n_2)}(\gamma) &= \{(i, j) \mid i < j \leq \gamma(i) \leq N_2 < \gamma(j) \text{ or } i > N_2 \geq j > \gamma(i) > \gamma(j)\}. \end{aligned}$$

Clearly, for any $\gamma \in \mathcal{S}_n^{(n_1, n_2)}$, we have $cr(\gamma) = \sum_{i=1}^5 |G_i^{(n_1, n_2)}(\gamma)|$. In particular,

$$cr(\sigma) = \sum_{i=1}^5 |G_i^{(n_1, n_2)}(\sigma)| \quad \text{and} \quad cr(\sigma') = \sum_{i=1}^5 |G_i^{(n_2, n_1)}(\sigma')|. \quad (47)$$

The "forms" of the crossings in the $G_i^{(n_1, n_2)}$'s and $G_i^{(n_2, n_1)}$'s are given in Table 5. By the definition of $\Gamma^{(n_1, n_2)}$, it is readily seen (see Row 1 in Table 6) that $G_1^{(n_1, n_2)}(\sigma) = G_1^{(n_2, n_1)}(\sigma')$ and thus $|G_1^{(n_1, n_2)}(\sigma)| = |G_1^{(n_2, n_1)}(\sigma')|$. By similar considerations we can prove (see Table 6) that $|G_i^{(n_1, n_2)}(\sigma)| = |G_i^{(n_2, n_1)}(\sigma')|$ for $i = 2, 3, 4$. It then suffices to prove that $|G_5^{(n_1, n_2)}(\sigma)| = |G_5^{(n_2, n_1)}(\sigma')|$ which will follow from the following lemma.

Lemma 13. *Let n_1, n_2, n be positive integers with $N_2 \leq n$ and $\gamma \in \mathcal{S}_n^{(n_1, n_2)}$. Suppose that*

$$\begin{aligned} B(\gamma) &:= \{(i, \gamma(i)) \mid i < \gamma(i) \leq N_2\} = \{(i_1, j_1), (i_2, j_2), \dots, (i_p, j_p)\} \\ B(\gamma^{-1}) &= \{(\gamma(i), i) \mid \gamma(i) < i \leq N_2\} = \{(k_1, \ell_1), (k_2, \ell_2), \dots, (k_q, \ell_q)\}, \end{aligned}$$

with $i_1 < i_2 < \dots < i_p$ and $k_1 < k_2 < \dots < k_q$. Then we have

$$|G_5^{(n_1, n_2)}(\gamma)| = \sum_{r=1}^p (j_r - i_r) + \sum_{r=1}^q (\ell_r - k_r - 1) - \binom{p+q}{2}. \quad (48)$$

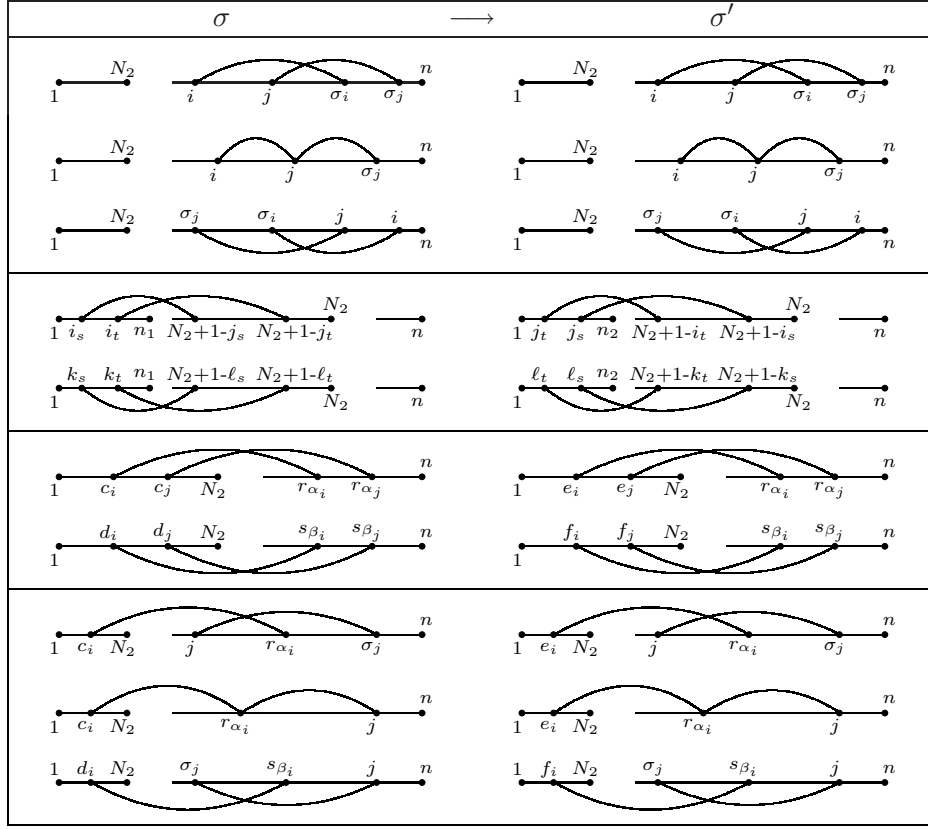
Suppose

$$\begin{aligned} B(\sigma) &= \{(i_1, N_2 + 1 - j_1), \dots, (i_p, N_2 + 1 - j_p)\} \\ B(\sigma^{-1}) &= \{(k_1, N_2 + 1 - \ell_1), \dots, (k_q, N_2 + 1 - \ell_q)\}, \end{aligned}$$

then, by construction of σ' , we have

$$\begin{aligned} B(\sigma') &= \{(j_1, N_2 + 1 - i_1), \dots, (j_p, N_2 + 1 - i_p)\} \\ B(\sigma'^{-1}) &= \{(\ell_1, N_2 + 1 - k_1), \dots, (\ell_q, N_2 + 1 - k_q)\}. \end{aligned}$$

By symmetry, the identity (48) is also valid on $\mathcal{S}_n^{(n_2, n_1)}$. Applying (48) to σ' and σ lead to $|G_5^{(n_1, n_2)}(\sigma)| = |G_5^{(n_2, n_1)}(\sigma')|$. This conclude the proof of Lemma 12. It then remains to prove Lemma 13.

TABLE 6. Effects of the mapping $\Gamma^{(n_1, n_2)}$ on the crossings of σ and σ' .

Proof of Lemma 13 By definition of $G_5^{(n_1, n_2)}(\gamma)$, we have

$$\begin{aligned}
 |G_5^{(n_1, n_2)}(\gamma)| &= |\{(i, j) \mid i < j < \gamma(i) \leq N_2 < \gamma(j)\}| + |\{(i, j) \mid \gamma(j) < \gamma(i) < j \leq N_2 < i\}| \\
 &\quad + |\{i \mid i < \gamma(i) \leq N_2 < \gamma^2(i)\}|.
 \end{aligned} \tag{49}$$

Now, by elementary manipulations and the definition of $B(\gamma)$ we get

$$\begin{aligned}
 |\{(i, j) \mid i < j < \gamma(i) \leq N_2 < \gamma(j)\}| &= \sum_{r=1}^p |\{x \mid i_r < x < j_r \leq N_2 < \gamma(x)\}| \\
 &= \sum_{r=1}^p |\{x \mid i_r < x < j_r\}| - |\{x \mid i_r < x < j_r, \gamma(x) \leq N_2\}|.
 \end{aligned}$$

But for any $r \in [1, p]$, we have $|\{x \mid i_r < x < j_r\}| = j_r - i_r - 1$ and

$$\begin{aligned} & |\{x \mid i_r < x < j_r, \gamma(x) \leq N_2\}| \\ &= |\{x \mid i_r < x < j_r, x < \gamma(x) \leq N_2\}| + |\{x \mid i_r < x < j_r, \gamma(x) < x \leq N_2\}| \\ &= |\{t \mid i_r < i_t < j_r\}| + |\{t \mid i_r < \ell_t < j_r\}| \quad \text{by definition of } B(\gamma) \text{ and } B(\gamma^{-1}), \\ &= |\{t \mid i_r < i_t\}| + |\{t \mid \ell_t < j_r\}|, \end{aligned}$$

since by definition of $\mathcal{S}_n^{(n_1, n_2)}$ we have that for any integers r and t , $i_r \leq n_1$, $k_t \leq n_1$, $j_r > n_1$ and $\ell_t > n_1$, and thus $i_r < j_t$ and $i_r < \ell_t$.

Summing over all r yields

$$|\{(i, j) \mid i < j < \gamma(i) \leq N_2 < \gamma(j)\}| = \sum_{r=1}^p (j_r - i_r - 1) - \sum_{r=1}^p |\{t \mid i_r < i_t\}| - \sum_{r=1}^p |\{t \mid \ell_t < j_r\}|. \quad (50)$$

Since $|\{(i, j) \mid \gamma(j) < \gamma(i) < j \leq N_2 < i\}| = |\{(i, j) \mid i < j < \gamma^{-1}(i) \leq N_2 < \gamma^{-1}(j)\}|$, it follows from (50) that

$$|\{(i, j) \mid \gamma(j) < \gamma(i) < j \leq N_2 < i\}| = \sum_{r=1}^q (\ell_r - k_r - 1) - \sum_{r=1}^q |\{t \mid k_r < k_t\}| - \sum_{r=1}^q |\{t \mid j_t < \ell_r\}|. \quad (51)$$

Remarking that $|\{i \mid i < \gamma(i) \leq N_2 < \gamma^2(i)\}| = |\{t \mid \gamma(j_t) > N_2\}|$ and inserting (50) and (51) in (49) lead to

$$\begin{aligned} |G_5^{(n_1, n_2)}(\gamma)| &= \sum_{r=1}^p (j_r - i_r - 1) + \sum_{r=1}^q (\ell_r - k_r - 1) + |\{t \mid \gamma(j_t) > N_2\}| - \sum_{r=1}^p |\{t \mid i_r < i_t\}| \\ &\quad - \sum_{r=1}^p |\{t \mid \ell_t < j_r\}| - \sum_{r=1}^q |\{t \mid k_r < k_t\}| - \sum_{r=1}^q |\{t \mid j_t < \ell_r\}|. \end{aligned} \quad (52)$$

Since the i_r 's and k_r 's are distinct we have

$$\sum_{r=1}^p |\{t \mid i_r < i_t\}| = \binom{p}{2} \quad \text{and} \quad \sum_{r=1}^q |\{t \mid k_r < k_t\}| = \binom{q}{2}. \quad (53)$$

On the other hand,

$$\begin{aligned} \sum_{r=1}^p |\{t \mid \ell_t < j_r\}| + \sum_{r=1}^q |\{t \mid j_t < \ell_r\}| &= \sum_{r=1}^p |\{t \mid \ell_t \neq j_r\}| \\ &= pq - |\{t \mid j_t \in \{\ell_1, \ell_2, \dots, \ell_q\}\}| \\ &= pq - \sum_{s=1}^k |\{t \mid \gamma(j_t) \leq N_2\}|, \end{aligned} \quad (54)$$

where the last identity follows from the definitions of $B(\gamma)$ and $B(\gamma^{-1})$. Inserting (53) and (54) in (52) lead to (48). This concludes the proof of Lemma 13. \square

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